

## Kink dynamics in the weakly stochastic $\phi^4$ model

Vladimir V. Konotop

*Institute for Radiophysics and Electronics, Ukrainian Soviet Socialist Republic Academy of Sciences,  
12 Proscura Street, Kharkov 310085, Ukraine, U.S.S.R.*

Angel Sánchez and Luis Vázquez

*Departamento de Física Teórica I, Universidad Complutense, Ciudad Universitaria, E-28040 Madrid, Spain  
(Received 24 October 1990)*

We describe how kink dynamics in a weakly stochastic, nonlinear  $\phi^4$  model can be understood by means of different analytical approaches. We first show how usual decompositions of the  $\phi^4$  chain excitations into translation, internal-oscillation, and phonon modes are valid up to a certain time that depends on the noise strength. By means of this decomposition we study the effects of dynamical disorder in the system on soliton propagation. The pinning of a kink on an unperturbed-perturbed boundary is considered in the adiabatic approximation. We then compute how the radiation generated by the kink absorbs energy in a resonant way from the noise at times later than the validity limit of our approximation. We conclude with a discussion of our findings and some comments on open questions and further extensions of this work.

### I. INTRODUCTION

In recent years, nonlinear science has received more and more attention from researchers from different fields and under different viewpoints, either theoretical, numerical, or experimental. The first two ways to study these problems are indeed closely related, and this relationship has been one of the most important reasons for the rapid development and achievement of new results.<sup>1</sup> No less important is the subsequent comparison of the properties of nonlinear models to the real systems that are supposedly described by them; though nonlinear experimental works are still not so numerous as analytical or numerical are, their importance is growing quite rapidly, mainly in condensed matter physics.<sup>2</sup> In fact, condensed matter physics is the field where nonlinear models have found the greatest number of applications in a very natural form, and also where the interaction between theory and numerics has been most fruitful.

Among the plethora of possible nonlinear models, nonlinear Klein-Gordon equations often arise in dealing with topics ranging from Josephson-junction theory to quasi-one-dimensional conductors or magnetic compounds.<sup>3,4</sup> Practically all these applications profit from the large amount of previous work on either the sine-Gordon equation or the  $\phi^4$  equation. While the former is integrable, the latter is not. In considering ideal systems, this is not so important, because both systems are rather well-known and analytical approaches are quite feasible. However, integrability makes a real difference when the problem of inhomogeneities, disorder, or unpredictable fluctuations of the system is taken into account. Theoretical tools for that kind of problems are currently available for the study of perturbations of the sine-Gordon and other integrable systems (see Ref. 5 for a complete review of

such techniques), but this is not so for  $\phi^4$  models, where methods are few and are, in many cases, particularly designed for each problem.

In this work we are concerned with a disordered version of the  $\phi^4$  model. The  $\phi^4$  model was first proposed by Aubry<sup>6</sup> and by Krumhansl and Schrieffer<sup>7</sup> to describe displacive and order-disorder transitions in solids, mainly magnetic compounds. Later on, Rice and Timonen<sup>8</sup> suggested that it could also serve as phenomenological model for polymers, and in fact it actually resembles the real system.<sup>9</sup> These pioneering works<sup>6,7</sup> studied both the dynamics and statistical mechanics of the  $\phi^4$  chain and showed that, in a good approximation, it could be considered as an ideal gas of kinks and phonons. This decomposition came out from the linearized equation for the chain excitations and its eigenfunctions, and became the most frequently used instrument in this problem, since similar works on the sine-Gordon model by Fogel *et al.*<sup>10</sup> and Eilenberger<sup>11</sup> The joint application of this procedure and the collective coordinates treatment of McLaughlin and Scott,<sup>12</sup> (see also Ref. 13) that considered kinks as point particles proved very fruitful. The next advances were done in the comprehension of the previously neglected kink-phonon interaction<sup>14</sup> and how is it related to the Brownian motion of kinks.<sup>15</sup> The concept of elementary excitations and mainly the particular role of the internal mode led Campbell, Schonfeld, and Wingate<sup>16</sup> to clearly establish and theoretically justify the previous<sup>17</sup> numerical hints on the existence of resonant long-lived, breatherlike structures in the  $\phi^4$  chain, showing once more the power of combining numerical and analytical approaches. Finally, while almost all of the above-mentioned researches work mainly using the continuum limit of the  $\phi^4$  model, the study of the intrinsically discrete effects was first (aside, of course, from nu-

merical calculations, see Ref. 18) carried out by Combs and co-workers,<sup>19</sup> who intended to improve the continuum treatment to explain pinning of solitons (found by Currie *et al.* in the related sine-Gordon system<sup>20</sup>) and some other problems concerning the true kink-phonon interaction.

Different forms of perturbations of the pure  $\phi^4$  model have been also studied in recent years because of its importance to comparison with real systems. Thus, Pnevmatikos *et al.* introduced<sup>21</sup> a number of generalizations in the chain, like external fields, dissipation, and nonlinear interparticle coupling, and studied kink stability and the emission of radiation, while Fraggis, Pnevmatikos, and Economou<sup>22</sup> considered the effect of substitutive impurities along the chain, and numerically showed that one of these was able to reflect a soliton or not, depending on the soliton speed. With respect to random perturbations, a very active field nowadays (a thorough review is given in Ref. 23), Collins and co-workers<sup>24</sup> and Abdullaev and co-workers<sup>25</sup> did some early work on additive Langevin-like forces acting on kinks and related their results to the dynamics of domain walls in ferroelectrics. Bass, Konotop, and Sinitsyn studied<sup>26</sup> the propagation of a  $\phi^4$  soliton through a three-dimensional random layer. They also computed<sup>27</sup> the probability distribution for the soliton collective coordinates in the sine-Gordon and  $\phi^4$  equations under the action of several multiplicative noises. Both additive and multiplicative stochastic terms were the subject of a recent paper by Rodríguez-Plaza and Vázquez<sup>28</sup> that extended the known results to the perturbatively unaccessible strong noise regime, and presented also analytical expressions for the probability distribution in some cases. Parrondo, Mañas, and de la Rubia<sup>29</sup> have gone a little further and obtained, in the context of a more general formalism, some predictions for nonlinear Klein-Gordon systems that are valid even for small correlation time colored noises. It is important to notice that a great amount of work is lacking in connection with randomly located inhomogeneities, which, as far as we know, have only been studied in the  $\phi^4$  chain for small-amplitude envelope solitons.<sup>30</sup>

The present paper is part of the current effort to understand the interaction of nonlinear waves with random fluctuations. Starting from some numerical results briefly reported in Ref. 31 (subsequently extended to other, more physical situations in Ref. 32), we have tried to achieve an analytical comprehension and to reproduce the features of the  $\phi^4$  model when the double well potential, after which the system is named, randomly changes its height in time. So, the quantities we are interested in will be those to be compared with the simulations, i.e., the kink “measurable” properties. In the next section we apply a perturbative formalism to describe the evolution of the kink for relatively short times, and we derive the expressions governing the appearance of radiation. In Sec.III we compute the contributions of each mode, recovering the equations for the collective coordinates. In Sec.IV, we discuss the physics of the corrections due to internal and phonon modes. Our perturbation theory allows us also to treat the pinning of a kink on an unperturbed-perturbed boundary that has been numeri-

cally observed;<sup>32</sup> this is shown in Sec. V. In Sec.VI we extend our study of the radiation to later times, by writing a linear equation for the excitations of the chain away the kink structure. By means of these calculations we learn how the radiation initially generated by the kink motion evolves by itself, growing in a resonant fashion so as to contribute to the exponential growing of the energy. At last, Sec.VII presents our main conclusions.

## II. SECULAR PERTURBATIVE THEORY

The particular model that we are concerned with in this paper was already presented in Ref. 31. In this model, soliton dynamics is governed by the following partial differential equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \epsilon_0 (\phi - \phi^3) = 0, \quad (1)$$

where  $\phi(x, t)$  stands for the displacement field, and  $\epsilon_0$  is the barrier height of the double well potential after which the system is named.<sup>6,7,31</sup> Fluctuations enter the model through  $\epsilon_0$ , in the form

$$\epsilon_0 \equiv 1 + \xi(t); \quad \langle \xi(t) \rangle = 0, \quad \langle \xi(t) \xi(t') \rangle = 2D \delta(t - t'), \quad (2)$$

the brackets having the usual meaning of mean values. Of course, were it not for the stochastic term in  $\epsilon_0$ , we would be left with the usual, well-known  $\phi^4$  equation.

The problem we are going to address is to describe how an initially unperturbed soliton with speed  $v$ , that solves Eq. (1) when the noise is switched off,  $D = 0$ , and whose expression is [with  $\gamma = (1 - v^2)^{-1/2}$ ],

$$\phi_v(x, t) = \tanh \left( \frac{\gamma}{\sqrt{2}} (x - vt - x_0) \right), \quad (3)$$

evolves under the influence of the fluctuating potential. In fact, as the formalism we are going to present is rather general, we will consider the problem

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \phi + \phi^3 = \varepsilon f(x, t) R[\phi], \quad (4)$$

with  $f(x, t) = 0$  if  $t < 0$ ; otherwise,  $f$  is any function,  $R$  is any functional of  $\phi$ , and  $\varepsilon$  is a small parameter that can be, e.g., the noise strength  $\sqrt{2D}$ .

What we will do now is to look for a solution of problem (4) in the form of an expansion in powers of  $\varepsilon$ , such as<sup>23</sup>

$$\phi(x, t) = \phi_0(x, t) + \varepsilon \phi_1(x, t) + \varepsilon^2 \phi_2(x, t) + \dots, \quad (5)$$

where  $\phi_0$  is a known exact solution of the unperturbed ( $\varepsilon = 0$ ) equation. We choose it to be a soliton with time dependent parameters of the form (c.f. the perturbation theory developed by McLaughlin and Scott<sup>12</sup>)

$$\phi_0(x, t) = \tanh \left( \frac{x - z(t, \tau)}{\sqrt{2[1 - v^2(\tau)]}} \right), \quad (6)$$

with

$$z(t, \tau) = x_0(\tau) + \int_0^t d\tau v(\tau), \quad (7)$$

and  $\tau = \varepsilon t$  is a slow variable defined to stress that  $dx_0/dt \simeq dv/dt \simeq \varepsilon$ . The reason for this choice is both physical and mathematical. Physically, it reflects the fact that the soliton collective coordinates can be affected by the noise; then, the dependence on  $\tau$  assumes that these changes in the kink motion, if any, are much more slow than the radiation generation, represented in the terms  $\phi_i(x, t)$ . Mathematically, it will become apparent later that we need these free functions, as it is usually done, to avoid secular terms in the perturbative expansion. We now proceed with the calculation. For the sake of convenience, let us split the second-order equation (4) into two first-order ones:

$$\phi_t(x, t) - \nu(x, t) = 0, \quad (8)$$

$$\nu_t(x, t) - \phi_{xx}(x, t) - \phi(x, t) + \phi^3(x, t) = \varepsilon f(x, t) R[\phi]. \quad (9)$$

Substitution of (5) [and a similar expansion for  $\nu(x, t)$ ] into (8) and (9), and subsequent comparison of  $\varepsilon$  orders, yields the first approximation equation (we drop for simplicity the explicit dependence on  $x$  and  $t$ ):

$$\phi_{1,t} - \nu_1 = -\phi_{0,\tau}, \quad (10)$$

$$\nu_{1,t} - \phi_{1,xx} - \phi_1 + 3\phi_0^2\phi_1 = f(x, t) R[\phi_0] - \nu_{0,\tau}. \quad (11)$$

Notice that the terms  $\phi_{0,\tau}$  and  $\nu_{0,\tau}$  come from the fact that  $\tau$  derivatives are one order  $\varepsilon$  higher than  $t$  derivatives. Correspondingly,  $t$  is considered a constant as one takes these derivatives (i.e.,  $\tau$  and  $t$  are assumed to be independent as usually in the multiscale method<sup>34</sup>), and

$$\frac{\partial}{\partial \tau} = \frac{\partial v}{\partial \tau} \frac{\partial}{\partial v} - \frac{\partial x_0}{\partial \tau} \frac{\partial}{\partial x}. \quad (12)$$

Next, we rewrite (10) and (11) in new variables, transforming to a reference frame moving with the nonlinear wave,

$$(t, x) \mapsto \left( t, \zeta \equiv \frac{x - z(t)}{\sqrt{2(1-v^2)}} \right), \quad (13)$$

which, within the desired accuracy, leads to

$$\phi_{1,t} - \frac{v}{\sqrt{2(1-v^2)}} \phi_{1,\zeta} - \nu_1 = -\phi_{0,\tau}, \quad (14)$$

$$\begin{aligned} \nu_{1,t} - \frac{v}{\sqrt{2(1-v^2)}} \nu_{1,\zeta} - \frac{1}{2(1-v^2)} \phi_{1,\zeta\zeta} \\ - \phi_1 + 3\phi_0^2(\zeta)\phi_1 = F(t, \zeta) R[\phi_0(\zeta)] - \nu_{0,\tau}, \end{aligned} \quad (15)$$

$F(t, \zeta)$  standing for  $f(t, x = z(t) + \zeta\sqrt{2(1-v^2)})$  and all the functions depending on  $(t, \zeta)$  unless otherwise indicated, like in  $\phi_0(\zeta)$ . Now we introduce the Fourier transform in time, defined by

$$\tilde{\phi}_1(\omega, \zeta) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i\omega t} \phi(t, \zeta). \quad (16)$$

We apply this transformation in equations (14) and (15), and after that, we obtain from (14) an expression for  $\tilde{\nu}(\omega, \zeta)$  that we can substitute in (15) to get

$$-\frac{1}{2}\tilde{\phi}_{\zeta\zeta} - \frac{2i\omega v}{\sqrt{2(1-v^2)}}\tilde{\phi}_{\zeta} - (1+\omega^2)\tilde{\phi} + 3\phi_0^2(\zeta)\tilde{\phi} = \frac{1}{2}\tilde{F}, \quad (17)$$

where we have dropped subindices 1, that from now on will be implicitly understood, and we have called

$$\begin{aligned} \tilde{F}(\omega, \zeta) \equiv 2\tilde{F}(\omega, \zeta) R[\phi_0(\zeta)] - 2\tilde{\nu}_{0,\tau} - 2i\omega\tilde{\phi}_{0,\tau} \\ + \frac{2v}{\sqrt{2(1-v^2)}}\tilde{\phi}_{0,\tau\zeta}. \end{aligned} \quad (18)$$

To exclude the first-order derivative in the left-hand side of Eq. (17), we look for a solution of it in the form

$$\tilde{\phi}(\omega, \zeta) \equiv \psi_\omega(\zeta) \exp\left(-i\frac{\sqrt{2}\omega v}{\sqrt{1-v^2}}\zeta\right). \quad (19)$$

Substituting this into Eq. (17), and recalling that  $\phi_0(\zeta) = \tanh \zeta$ , we find

$$\begin{aligned} \ddot{\psi} + \left(\frac{2\omega^2}{1-v^2} - 4 + \frac{6}{\cosh^2 \zeta}\right) \psi \\ = -\tilde{F}(\omega, \zeta) \exp\left(i\frac{\sqrt{2}\omega v}{\sqrt{1-v^2}}\zeta\right) \equiv \tilde{\Gamma}(\omega, \zeta), \end{aligned} \quad (20)$$

dots meaning derivative with respect to  $\zeta$ . So, we have to deal with a well-known<sup>7,33,34</sup> Schrödinger-like eigenvalue problem for the family of operators

$$L_\omega \equiv \frac{d^2}{d\zeta^2} + E(\omega) + \frac{6}{\cosh^2(\zeta)}, \quad E(\omega) \equiv \frac{2\omega^2}{1-v^2} - 4. \quad (21)$$

The spectrum of  $L_\omega$  is given by two discrete eigenvalues,  $\lambda_0 = E(\omega) + 4$ ,  $\lambda_1 = E(\omega) + 1$  with associated normalized eigenfunctions

$$\psi_0(\zeta) = \sqrt{\frac{3}{4}} \frac{1}{\cosh^2(\zeta)}, \quad \psi_1(\zeta) = \sqrt{\frac{3}{2}} \frac{\sinh(\zeta)}{\cosh^2(\zeta)}, \quad (22)$$

and a continuum  $\lambda_k = E(\omega) - k^2$ , its respective eigenfunctions being

$$\psi_k(\zeta) = e^{ik\zeta} \frac{3 \tanh^2 \zeta - 3ik \tanh \zeta - (1+k^2)}{[2\pi(1+k^2)(4+k^2)]^{1/2}}. \quad (23)$$

As this functions form an orthonormal basis, we can directly write down an expression for the function  $\psi(\zeta)$  we are looking for

$$\psi_\omega(\zeta) = \frac{\gamma_0(\omega)\psi_0(\zeta)}{E(\omega) + 4} + \frac{\gamma_1(\omega)\psi_1(\zeta)}{E(\omega) + 1} + \int_{-\infty}^{\infty} dk \frac{\gamma_k(\omega)\psi_k(\zeta)}{E(\omega) - k^2}, \quad (24)$$

with

$$\gamma_i(\omega) \equiv \int_{-\infty}^{\infty} d\zeta \tilde{\Gamma}(\omega, \zeta) \psi_i^*(\zeta) \quad (25)$$

being the coefficients of the orthonormal decomposition of the source term  $\tilde{\Gamma}(\omega, \zeta)$  (asterisk stands for complex conjugation). This is the final result of the calculations for the first order approximation; what is left in principle is only algebra, namely, to compute the integrals in (25) and once that is done, the first correction is given by

$$\phi(t, \zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \exp\left(-i \frac{\sqrt{2}\omega v}{\sqrt{1-v^2}}\right) \psi_\omega(\zeta). \quad (26)$$

The point is that in our particular calculations we have the problem that, hidden in the function  $\tilde{\Gamma}(\omega, \zeta)$  through the sequence of changes

$$f(x, t) \xrightarrow{x \mapsto \zeta} F(t, \zeta) \xrightarrow{FT} \tilde{F}(\omega, \zeta) \xrightarrow{\text{def}} \tilde{\mathcal{F}}(\omega, \zeta) \xrightarrow{\text{def}} \Gamma(\omega, \zeta), \quad (27)$$

there is a stochastic process,  $f(x, t) = \xi(t)$ , and consequently the calculation of the coefficients  $\gamma_0$ ,  $\gamma_1$ , and  $\gamma_k$  in an evident form seems impossible. In spite of this problem, we are going to show below that some useful information can still be obtained from the previous perturbative formalism.

### III. MODES CONTRIBUTIONS

Let us rewrite Eq. (26) for the first-order correction to the unperturbed kink with time dependent parameters. Taking into account the three different contributions in  $\psi(\zeta)$  in (24), namely, the ones from the zero (translation or Goldstone) mode, the internal (localized) mode, and the continuum (radiation or phonon) modes, we can divide also the expression (26) for  $\phi$  into three different contributions:

$$\phi(t, \zeta) = \phi^{(0)}(t, \zeta) + \phi^{(1)}(t, \zeta) + \phi^{(\text{ph})}(t, \zeta), \quad (28)$$

where we have defined

$$\phi^{(0)}(t, \zeta) = -\frac{1}{\sqrt{2\pi}} \psi_0(\zeta) \int_{-\infty}^{\infty} d\omega e^{i\omega t} \frac{\exp[-i\zeta(\sqrt{2}\omega v/\sqrt{1-v^2})]}{E(\omega) + 4} \int_{-\infty}^{\infty} d\zeta' \psi_0(\zeta') \tilde{\mathcal{F}}(\omega, \zeta') \exp\left(i\zeta' \frac{\sqrt{2}\omega v}{\sqrt{1-v^2}}\right), \quad (29)$$

$$\phi^{(1)}(t, \zeta) = -\frac{1}{\sqrt{2\pi}} \psi_1(\zeta) \int_{-\infty}^{\infty} d\omega e^{i\omega t} \frac{\exp[-i\zeta(\sqrt{2}\omega v/\sqrt{1-v^2})]}{E(\omega) + 1} \int_{-\infty}^{\infty} d\zeta' \psi_1(\zeta') \tilde{\mathcal{F}}(\omega, \zeta') \exp\left(i\zeta' \frac{\sqrt{2}\omega v}{\sqrt{1-v^2}}\right), \quad (30)$$

$$\phi^{(\text{ph})}(t, \zeta) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \psi_k(\zeta) \int_{-\infty}^{\infty} d\omega e^{i\omega t} \frac{\exp[-i\zeta(\sqrt{2}\omega v/\sqrt{1-v^2})]}{E(\omega) - k^2} \int_{-\infty}^{\infty} d\zeta' \psi_k^*(\zeta') \tilde{\mathcal{F}}(\omega, \zeta') \exp\left(i\zeta' \frac{\sqrt{2}\omega v}{\sqrt{1-v^2}}\right). \quad (31)$$

We will deal with these separate contributions term by term. On the first hand, we compute  $\phi^{(0)}(x, t)$ . Substitution of  $\tilde{\mathcal{F}}$  by its expression (18) and inversion of Fourier transforms yield

$$\begin{aligned} \phi^{(0)}(t, \zeta) = -\frac{1-v^2}{2\pi} \psi_0(\zeta) \left\{ \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} \int_{-\infty}^{\infty} d\zeta' \int_{-\infty}^{\infty} dt' e^{i\omega\mu} \psi_0(\zeta') \left[ F(t', \zeta') R[\phi_0(\zeta')] - \left( \nu_{0,\tau} - \frac{v}{\sqrt{2(1-v^2)}} \phi_{0,\tau\zeta'} \right) \right] \right. \\ \left. - i \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \int_{-\infty}^{\infty} d\zeta' \int_{-\infty}^{\infty} dt' e^{i\omega\mu} \psi_0(\zeta') \phi_{0,\tau} \right\}, \end{aligned} \quad (32)$$

where we have put

$$\mu \equiv (t - t') - \frac{\sqrt{2}v}{\sqrt{1-v^2}}(\zeta - \zeta'). \quad (33)$$

The integrals over  $\omega$  can be easily computed in the complex plane, and (32) becomes

$$\phi^{(0)}(t, \zeta) = (1-v^2) \psi_0(\zeta) \int_{-\infty}^{\infty} d\zeta' \int_{-\infty}^{\infty} dt' \psi_0(\zeta') \theta(\mu) \left[ \mu \left( F(t', \zeta') R[\phi_0(\zeta')] + \frac{v}{\sqrt{2(1-v^2)}} \phi_{0,\tau\zeta'} - \nu_{0,\tau} \right) - \phi_{0,\tau} \right], \quad (34)$$

$\Theta(\mu)$  being the Heaviside step function. From this equation it can be seen that, as the integrals have an upper limit given by  $t - \sqrt{2}v(\zeta - \zeta')/\sqrt{1-v^2}$ , which comes from  $\theta(\mu)$ ,  $\phi^{(0)}$  will indefinitely grow when  $t \rightarrow \infty$ , a secular term thus

arising. Then, we must request that this contribution vanishes; otherwise, our perturbation theory to compute small corrections would be totally unjustified (it would be necessary to restrict it to very small times). After calculating  $\nu_{0,\tau}$ ,  $\phi_{0,\tau}$ , and  $\phi_{0,\tau\zeta'}$  and some straightforward algebra, it is not difficult to show that the following equations (the so-called adiabatic approximation)

$$\frac{dv}{dt} = -\frac{3}{4}\sqrt{2} [1 - v^2(t)]^{3/2} \varepsilon \int_{-\infty}^{\infty} \frac{d\zeta'}{\cosh^2 \zeta'} F(t, \zeta') R[\phi_0(\zeta')], \quad (35)$$

$$\frac{dz}{dt} = v(t) - \frac{3}{2}v(t) [1 - v^2(t)] \varepsilon \int_{-\infty}^{\infty} \frac{d\zeta'}{\cosh^2 \zeta'} \zeta' F(t, \zeta') R[\phi_0(\zeta')] \quad (36)$$

must be verified if the secular term is not to appear in our expansion. Here, we have returned from the slow variable  $\tau$  to  $t$ . Note that the equations of the adiabatic approximation can be obtained also as an orthogonality condition<sup>12,27</sup> for the absence of secular terms.

We can now recall the precise form of our perturbation and particularize (35) and (36), to arrive at last to

$$v'(t) = 0, \quad (37)$$

$$z'(t) = v(t) - \frac{1}{2}v(t) [1 - v^2(t)] \xi(t). \quad (38)$$

Had we considered dissipation terms, we would have gotten

$$v'(t) = -\alpha\gamma v(t), \quad (39)$$

the same equation in (38) still holding for  $z(t)$ . It is now clear that these collective coordinate differential equations are a consequence of the translation mode contributions, and that their derivation through conserved quantities<sup>5</sup> involves a neglecting of the role of the internal mode and the phonon continuum in the kink evolution when a noise term acts. Finally, together with the equations of the adiabatic approximation, formula (34) for the zero mode contribution becomes

$$\phi^{(0)}(t, \zeta) = (1 - v^2) \psi_0(\zeta) \int_{-\infty}^{\infty} d\zeta' \psi_0(\zeta') \int_{-\infty}^{\infty} dt' \theta(\mu) \int_{-\infty}^{\infty} d\zeta'' F(t', \zeta'') R[\phi_0(\zeta'')] [\mu A(\zeta', \zeta''; v) - B(\zeta', \zeta''; v)], \quad (40)$$

where

$$A(\zeta', \zeta''; v) = \delta(\zeta' - \zeta'') + \frac{3v^2}{2} \phi_{0,\zeta'\zeta'} \frac{\zeta'' - \zeta'}{\cosh^2 \zeta''} - \frac{3}{4}(1 + v^2) \phi_{0,\zeta'} \frac{1}{\cosh^2 \zeta''}, \quad (41)$$

and

$$B(\zeta', \zeta''; v) = \frac{3v\sqrt{1-v^2}}{2\sqrt{2}} \frac{\zeta'' - \zeta'}{\cosh^2 \zeta''} \phi_{0,\zeta'}. \quad (42)$$

If we now particularize for the case under consideration, it is not difficult to show that  $\phi^{(0)}$  vanishes. This means that the adiabatic approximation takes into account all the perturbation effects related to the zero mode, and it is a consequence of the spatial homogeneity of the perturbation. This statement, together with Eqs. (37) and (38), leads us to the conclusion that changes of the kink shape, if any, are completely described by the first mode only (let us note that an analogous situation for the sine-Gordon system is well known:<sup>5</sup> it can be proved that there is not any localized addendum to the perturbed solution because the only discrete eigenvalue is that of the zero mode).

We next go on and compute the internal mode contribution. Calculations are essentially the same as in the previous case, and only simple complex plane integrations, a certain amount of algebra, and the use of Eqs. (35) and (36) are required to write down the following formula for the localized mode contribution:

$$\begin{aligned} \phi^{(1)}(t, \zeta) = & (1 - v^2) \psi_1(\zeta) \int_{-\infty}^{\infty} d\zeta' \psi_1(\zeta') \int_{-\infty}^{\infty} dt' \theta(\mu) \int_{-\infty}^{\infty} d\zeta'' F(t', \zeta'') R[\phi_0(\zeta'')] \\ & \times \left( \frac{\sin(\Omega\mu)}{\Omega} A(\zeta', \zeta''; v) - \cos(\Omega\mu) B(\zeta', \zeta''; v) \right) \end{aligned} \quad (43)$$

where  $\Omega \equiv \sqrt{\frac{3}{2}(1 - v^2)}$ . This formula reduces to

$$\phi^{(1)}(t, \zeta) = \frac{3}{4} (1 - v^2) \frac{\sinh \zeta}{\cosh^2 \zeta} \int_{-\infty}^{\infty} d\zeta' \frac{\sinh \zeta'}{\cosh^4 \zeta'} \int_0^{\infty} dt' \theta(\mu) \xi(t') \left( 2(1 - v^2) \frac{\sin(\Omega\mu)}{\Omega} \tanh \zeta' - \frac{v\sqrt{1-v^2}}{\sqrt{2}} \cos(\Omega\mu) \right) \quad (44)$$

when particularized to our problem  $\varepsilon f(x, t) R[\phi] = \xi(t)(\phi - \phi^3)$ . With respect to the phonon contribution, it is given by

$$\begin{aligned} \phi^{(\text{ph})}(t, \zeta) = (1 - v^2) \int_{-\infty}^{\infty} dk \psi_k(\zeta) \int_{-\infty}^{\infty} d\zeta' \psi_k^*(\zeta') \\ \times \int_{-\infty}^{\infty} dt' \theta(\mu) \int_{-\infty}^{\infty} d\zeta'' F(t', \zeta'') R[\phi_0(\zeta'')] \\ \times \left( \frac{\sin(\Omega_k \mu)}{\Omega_k} A(\zeta', \zeta'', v) - \cos(\Omega_k \mu) B(\zeta', \zeta'', v) \right) \end{aligned} \quad (45)$$

with

$$\Omega_k^2 \equiv \frac{(k^2 + 4)}{2} (1 - v^2) \quad (46)$$

this time. Again, after particularization, (45) becomes

$$\begin{aligned} \phi^{(\text{ph})}(t, \zeta) = (1 - v^2) \int_{-\infty}^{\infty} dk \psi_k(\zeta) \int_{-\infty}^{\infty} d\zeta' \psi_k^*(\zeta') \int_0^{\infty} dt' \theta(\mu) \xi(t') \\ \times \frac{1}{\cosh^2 \zeta'} \left( 2(1 - v^2) \frac{\sin(\Omega_k \mu)}{\Omega_k} \tanh \zeta' - \frac{v\sqrt{1-v^2}}{\sqrt{2}} \cos(\Omega_k \mu) \right). \end{aligned} \quad (47)$$

#### IV. DISCUSSION

As we have obtained rather evident expressions for the different mode contributions, we are now able to make some general remarks concerning the stochastic  $\phi^4$  model. First of all, a simple and natural physical result is that all  $\phi^{(i)}(t, \zeta)$  are linear functionals of the random field  $\xi(t)$ , and subsequently their corresponding mean values are equal to zero,  $\langle \phi^{(i)}(t, \zeta) \rangle = 0$ . This is the reason why their statistics are completely determined by second-order momenta. On the other hand, as it follows from the factor  $(1 - v^2)$ , which appears in front of all integrals in Eqs. (44) and (47), an ultrarelativistic kink with  $v \sim 1$  is not essentially affected by noise. Note that this conclusion also arises from numerical simulations,<sup>31,32</sup> where we observed that perturbation effects get weaker as we approached the maximum allowed speed.

When all modes give an additive contribution [see Eq. (28)], we can consider them separately. The statistical properties of the kink velocity, as described by the adiabatic approximation in (38) are rather simple, and they have been discussed elsewhere.<sup>32</sup> We now concentrate our attention on both the internal and phonon modes; moreover, to make our examination more easily understandable, we restrict ourselves to kinks at rest, i.e., we put  $v = 0$  in all calculations in this section.

##### A. Internal mode

Let us compute the mean energy density of the internal mode. To this end we first rewrite Eq. (44) for our choice  $v = 0$ :

$$\phi^{(1)}(t, \zeta) = \frac{\sqrt{6}}{64} \pi \frac{\sinh \zeta}{\cosh^2 \zeta} \int_0^t dt' \xi(t') \sin[\sqrt{3/2}(t - t')]. \quad (48)$$

From this equation it becomes evident that the internal mode does not affect the dynamics of the soliton center and it takes part only in the fluctuations of the kink width. If we recall Eq. (38), we can conclude that the soliton velocity is not affected by noise if its initial value is equal to zero. Again, this is in perfect agreement with simulations.

The mean value of the energy density follows from Eq. (48) after direct averaging of the squares of both sides of it. This yields

$$\langle \phi^{(1)2}(t, \zeta) \rangle = \frac{3\pi^2}{2^{10}} D \frac{\sinh^2 \zeta}{\cosh^4 \zeta} \left( t - \frac{1}{\sqrt{6}} \sin \sqrt{6} t \right). \quad (49)$$

Thus, the energy of the internal mode grows with time according to the diffusion law, i.e.,  $\sim D t$ . In other words, an additional (we mean in comparison with a sine-Gordon kink) internal degree of freedom undergoes the usual diffusion caused by a linear Gaussian random force. The function  $\phi^{(1)}(t, \zeta)$  plays then the role of a coordinate in this treatment. However, we have not observed any visible changing of the kink shape in our numerical studies<sup>31,32</sup> for weak noises, the range to which perturbative formalism applies. We believe that this fact can be

explained by the small numerical prefactor in Eq. (49), that makes the effective diffusion coefficient about one order of magnitude smaller than  $D$ . We also feel that the behavior here described repeats its main features for nonzero speed (but nonrelativistic) kinks, the main difference being the smaller strength of the perturbation effects due to the speed dependent factor.

### B. Radiation

We now start the examination of the contributions from phonon modes by rewriting Eq. (47) in a more convenient (or more physical) fashion. To do that, we take into account the definitions (33) and (46), to present  $\phi^{(\text{ph})}(t, \zeta)$  in the form of an expansion in linear plane waves:

$$\phi^{(\text{ph})}(t, \zeta) = \int_{-\infty}^{\infty} dk [A_+(k) e^{i(K+\zeta-\Omega_+ t)} + A_-(k) e^{i(K-\zeta-\Omega_- t)}]. \quad (50)$$

Here we have put

$$K_{\pm} \equiv \frac{k \mp v\sqrt{k^2 + 4}}{\sqrt{2(1-v^2)}} \quad (51)$$

$$\Omega_{\pm} \equiv \frac{kv \mp \sqrt{k^2 + 4}}{\sqrt{2(1-v^2)}} \quad (52)$$

$$A_{\pm} \equiv - (1-v^2)^{3/2} \frac{3 \tanh^2 \zeta - 3ik \tanh \zeta - (1+k^2)}{[2\pi(1+k^2)(4+k^2)]^{1/2}} \times \int_{-\infty}^{\infty} d\zeta' \psi_k^*(\zeta') \int_0^{\infty} dt' \theta(\mu) \xi(t') e^{\mp i\Omega_{\pm}(t' - \sqrt{2}v\zeta'/\sqrt{1-v^2})} \left( \pm \sqrt{\frac{2}{3}} i \tanh \zeta' + \frac{v}{2\sqrt{2}} \right) \frac{1}{\cosh^2 \zeta'}. \quad (53)$$

Thus, the parameters  $K_{\pm}$  and  $\Omega_{\pm}$  have the sense of wave vectors and frequencies of radiation, respectively. It is not difficult to verify that they satisfy the dispersion relation

$$\Omega_{\pm}^2 = K_{\pm}^2 + 2, \quad (54)$$

i.e., the dispersion relation of linear modes overimposed to a kink.<sup>7</sup> The functions  $A_{\pm}(k)$  play the role of the radiation spectrum and describe the amplitude of linear noise generated in the system.

Radiation (or phonon modes) gives the most essential contribution to the total solution in the region of the kink tails, far away from the center. That is why we now study  $A_{\pm}(k)$  at  $|\zeta| \gg 1$ . Taking in addition  $v = 0$  as in Sec. IV A we can simplify the functions arriving at

$$A_{\pm}(k) = \pm a(k) \int_0^t dt' \xi(t') e^{\mp i\sqrt{3/2}t'}, \quad (55)$$

where the function

$$a(k) = \sqrt{\frac{2}{3}} k \frac{2 - 3ik \operatorname{sgn} \zeta - k^2}{(1+k^2)(4+k^2)} \quad (56)$$

contains all the dependence of the spectrum on the parameter  $k$ . The mean phonon energy density follows easily; indeed, using Eq. (55) and using the noise statistics (2) we find that

$$\langle |A_{\pm}|^2 \rangle = \frac{4}{3} \frac{k^2(k^4 + 5k^2 + 4)}{(1+k^2)^2(4+k^2)^2} D t. \quad (57)$$

The  $k$  dependence of this energy density is depicted in Fig. 1. From this picture one can see that the greatest amount of radiation is generated with parameters

$$k_{\max} \approx 1.425 \quad (58)$$

and directed towards both sides of the kink struc-

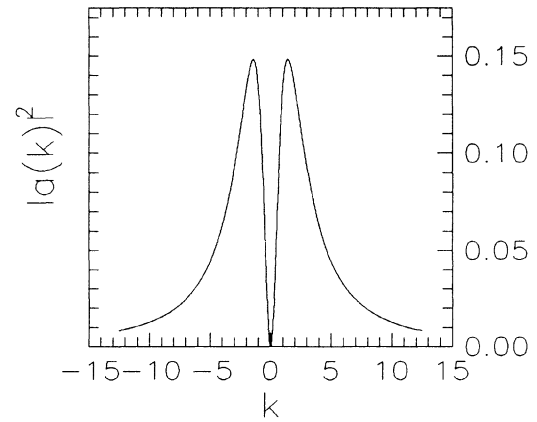


FIG. 1. Mean spectral density of the radiation energy. The tails of the spectrum have a simple analytical form,  $\langle |a(k)|^2 \rangle \propto k^{-2}$ . (The values of maxima are given in the text.)

ture. These values correspond to linear waves having wavenumbers  $K_+ = K_- = k_{\max}/\sqrt{2} \approx 1.008$  and frequencies  $\Omega_{\pm} = \mp \sqrt{(k_{\max}^2 + 4)/2} \approx 1.736$ .

The curve in Fig. 1 is symmetrical; besides, it behaves as  $4k^2$  in the neighborhood of  $k = 0$  and as  $k^{-2}$  as  $k \rightarrow \pm\infty$ . Moreover, result (57) does not depend on  $\text{sgn}\zeta$ , this meaning that the intensity of waves generated in both directions are the same. It is obvious that this fact results from the initial restriction to null velocity of the kink, and nonzero values of the velocity would lead to symmetry breaking. The corresponding laws can be obtained in the same way as for kinks at rest, involving only more complicated algebra to get rid of Eq. (53).

As we have shown previously, we have obtained a diffusive growth of the mean energy [see (57)]. It has the following physical interpretation: the derivative  $d(|A_{\pm}|^2)/dt$  has the sense of a power density of the emitted radiation. The potential fluctuations act on the kink continuously, i.e., phonons are being created steadily. Then formula (57) means that the power of this process of phonon generation does not depend on time.

## V. INTERACTION OF A KINK WITH AN UNPERTURBED-PERTURBED BOUNDARY

As we have learned from simulations,<sup>32</sup> when only a half of the space is under the effect of potential fluctuations, kink dynamics essentially differs from the case in which randomness extends to the whole spatial axis. Namely, a kink reaching an unperturbed-perturbed boundary is rapidly pinned at it if its initial velocity is small enough. In spite of the fact that the corresponding mathematical problem allows us to introduce a Fokker-Planck equation for the probability density of the soliton parameters in the adiabatic approximation, it seems to us impossible to calculate all the statistical characteristics in a useful form. So, we restrict ourselves to the demonstration of the following statement: the mean velocity of a kink scattered by a boundary between unperturbed and perturbed parts of the space decreases if its initial value is sufficiently small. Mathematically, this means that we now consider the problem (1) with

$$\epsilon_0 \equiv 1 + \theta(x)\xi(t), \quad (59)$$

$\xi(t)$  being once more a Gaussian white noise with statistics given by Eqs. (2). Thus, in terms of Eq. (4), we have a new function  $f(x, t) \equiv \theta(x)\xi(t)$ , the boundary being located at  $x = 0$  without loss of generality. Inserting these functions into the equations of the adiabatic approximation (35) and (36) and computing the resulting integrals, we can write down the system governing the evolution of the soliton parameters:

$$\frac{dv}{dt} = -\frac{3}{8} (1 - v^2)^{3/2} \xi(t) a_v(z, v), \quad (60)$$

$$\frac{dz}{dt} = v - \frac{3}{8} v (1 - v^2) \xi(t) a_z(z, v), \quad (61)$$

with

$$a_v(z, v) \equiv \frac{1}{\sqrt{2}} \cosh^{-4} \left( \frac{z}{\sqrt{2(1-v^2)}} \right), \quad (62)$$

$$a_z(z, v) \equiv -\frac{z}{\sqrt{2(1-v^2)} \cosh^4[z/\sqrt{2(1-v^2)}]} + \frac{2}{3} + \tanh \left( \frac{z}{\sqrt{2(1-v^2)}} \right) - \frac{1}{3} \tanh^3 \left( \frac{z}{\sqrt{2(1-v^2)}} \right). \quad (63)$$

For the following discussion, it is much more convenient to change into new variables

$$u \equiv \frac{v}{\sqrt{1-v^2}}, \quad \zeta \equiv \frac{z}{\sqrt{1-v^2}}; \quad (64)$$

then,  $a_z(z, v) = a_z(\zeta)$ ,  $a_v(z, v) = a_v(\zeta)$ , and Eqs. (60) and (61) transform into

$$\frac{du}{dt} = -\hat{\xi}(t) a_v(\zeta), \quad (65)$$

$$\frac{d\zeta}{dt} = u - \hat{\xi}(t) \frac{u}{1+u^2} \mathcal{A}(\zeta). \quad (66)$$

Here,

$$\mathcal{A}(\zeta) \equiv a_z(\zeta) + \frac{\zeta}{\sqrt{2}} a_v(\zeta), \quad (67)$$

and  $\hat{\xi}(t) \equiv \frac{3}{8} \xi(t)$  is a renormalized random function (correspondingly, below we will write  $\hat{D} = \frac{9}{64} D$ ).

Let us compute  $\langle d\zeta/dt \rangle$ . In order to do it, we will recall a celebrated theorem by Novikov,<sup>35</sup> (see also Ref. 36) that states

$$\langle \xi(t) \mathcal{F}[\xi] \rangle = \int_0^t dt' \langle \xi(t) \xi(t') \rangle \left\langle \frac{\delta \mathcal{F}[\xi]}{\delta \xi(t')} \right\rangle, \quad (68)$$

$\mathcal{F}[\xi]$  being any functional of the stochastic process  $\xi(t)$ . With the help of this expression, we can direct average (65) and (66), obtaining

$$\left\langle \frac{du}{dt} \right\rangle = -\frac{4\hat{D}}{\sqrt{2}} \left\langle \frac{u}{1+u^2} \frac{\sinh(\zeta/\sqrt{2})}{\cosh^5(\zeta/\sqrt{2})} \mathcal{A}(\zeta) \right\rangle, \quad (69)$$

$$\left\langle \frac{d\zeta}{dt} \right\rangle = \langle u \rangle + \hat{D} \left\langle \frac{1+u^4}{(1+u^2)^2} \frac{\mathcal{A}(\zeta)}{\sqrt{2} \cosh^4(\zeta/\sqrt{2})} \right\rangle. \quad (70)$$

The pinning of a kink colliding with an unperturbed-perturbed boundary takes place only if the initial kink speed is sufficiently small,<sup>32</sup> and that is what we want to prove. On the other hand, we must have in mind that we are considering the problem in the framework of a perturbation theory, i.e., when  $\hat{D} \ll 1$ . In view of this, it is natural to look for the effects happening in the region of very small velocities  $v \ll 1$  [which corresponds to  $u \ll 1$ , according to definition (64)] because these will be the ones affected by the weak noises to which our theory is restricted. More precisely, we will assume the dispersion of the velocity fluctuations to be of order  $\hat{D}$ . Under this condition the expressions in angular brackets in (69) and (70) can be simplified by neglecting terms of order  $u^2$  and higher, and in this way we are left with

$$\left\langle \frac{du}{dt} \right\rangle = -4\hat{D} \left\langle u \tanh \left( \frac{\zeta}{\sqrt{2}} \right) \mathcal{A}(\zeta) a_v(\zeta) \right\rangle, \quad (71)$$



$$\left\langle \frac{d\zeta}{dt} \right\rangle = \langle u \rangle + \hat{D} \langle \mathcal{A}(\zeta) a_v(\zeta) \rangle, \quad (72)$$

where terms of order  $\hat{D}^4 \sim u^2 \hat{D}^2$  have been omitted.

In order to solve the system (71) and (72), we must find  $\langle u \rangle$  and insert it into the equation for  $d\langle \zeta \rangle/dt$ . The problem is that the unknown  $u$  appears in the right-hand side (rhs) of Eq. (71). To remove this difficulty, we again use the restriction  $u \ll 1$  to linearize Eq. (66). This yields

$$u = \frac{du}{dt} [1 + \hat{\xi}(t) \mathcal{A}(\zeta)], \quad (73)$$

an expression for  $u$  that we can now insert into Eq. (71), and so, up to leading order, we have

$$\frac{du}{dt} = -4 \hat{D} \frac{d}{dt} \left\langle F \left( \frac{\zeta}{\sqrt{2}} \right) \right\rangle, \quad (74)$$

where

$$F(x) \equiv -\frac{1}{6 \cosh^4 x} + \frac{6}{35} \tanh x - \frac{2}{35} \tanh^3 x - \frac{13}{105} \frac{\sinh x}{\cosh^5 x} - \frac{1}{21} \frac{\sinh x}{\cosh^7 x}. \quad (75)$$

Taking into account that

$$\langle u \rangle(t=0) = u_0 = \frac{v_0}{\sqrt{1-v_0^2}} \approx v_0, \quad (76)$$

where  $0 < v_0 \ll 1$  is the initial velocity of the kink directed to the boundary, Eq. (74) can easily be integrated. If, for example, we assume in addition that the initial position of the kink center coincides with the origin, the outcome can be expressed as

$$\langle u \rangle = u_0 - \frac{2}{3} \hat{D} - 4 \hat{D} \left\langle F \left( \frac{\zeta}{\sqrt{2}} \right) \right\rangle. \quad (77)$$

Of course, now we are able to compute  $d\langle z \rangle/dt$  from Eq. (72). From the definition of  $\zeta$  in Eq. (64), we have

$$\left\langle \frac{d\zeta}{dt} \right\rangle \approx \left\langle \frac{dz}{dt} \right\rangle + \left\langle uz \frac{du}{dt} \right\rangle. \quad (78)$$

Combining Eqs. (65), (72), (77), and (78), and applying again to this last one the Novikov theorem (68) to deal with the last item in its rhs, we arrive at the final expression, namely,

$$\left\langle \frac{dz}{dt} \right\rangle = \hat{D} \left\langle \Phi \left( \frac{\zeta}{\sqrt{2}} \right) \right\rangle + u_0 - \frac{2}{3} \hat{D}, \quad (79)$$

with

$$\Phi(x) \equiv \frac{4}{3} \frac{1}{\cosh^4 x} - \frac{24}{35} \tanh x + \frac{8}{35} \tanh^3 x + \frac{122}{105} \frac{\sinh x}{\cosh^5 x} + \frac{11}{21} \frac{\sinh x}{\cosh^7 x} - \frac{x}{\sqrt{2} \cosh^8 x}. \quad (80)$$

Let us analyze the function  $\Phi(x)$  in more detail (see plot in Fig. 2). The asymptotic behaviors of it are  $\Phi(\pm\infty) = \mp \frac{16}{35}$ ; these values are approached exponentially at  $x < -1$  and  $x > 1$ , respectively. Note that  $\Phi(0) = \frac{4}{3}$  and  $\Phi'(0) > 0$ , and hence  $\max \Phi(x) \equiv \Phi_{\max}$  is

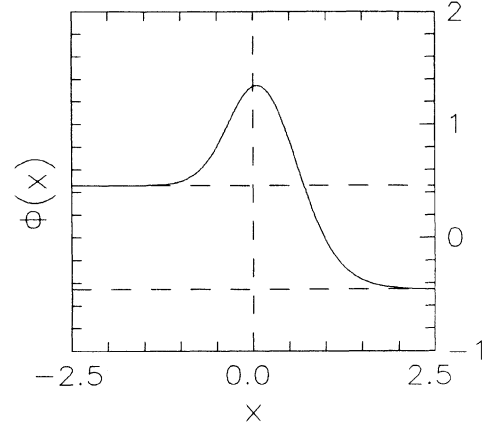


FIG. 2. The function  $\Phi(x)$  appearing in Eq. (79).

at  $x = x_{\max} > 0$ ; numerically it can be computed that  $\Phi_{\max} \approx 1.341$  and  $x_{\max} \approx 0.0526$ . We must now take into account two facts: first, as we have previously assumed, the dispersion of fluctuations of both  $v(t)$  and  $z(t)$  are of order  $\hat{D}t$  and second, a position  $z \lesssim 1$  can be reached by the kink at times around  $t \lesssim \hat{D}^{-1}$ , since  $u_0 \sim \hat{D}$  and  $x_0$  is taken to be zero (this is not a restriction because we can always redefine  $t$  taking as the origin of time for the moment at which the kink kernel just touches the boundary). From these two features and the form of the function  $\Phi$  it turns out by simple inspection that the averaged value in the rhs of Eq.(79) must take a zero value at some time  $t \lesssim \hat{D}^{-1}$  if  $u_0 \lesssim \hat{D}$ . When this happens, the kink gets pinned at the boundary or, more precisely, at the fluctuating region near the boundary, as we wanted to show.

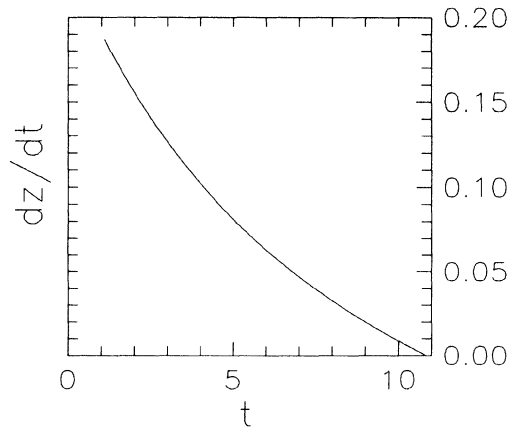


FIG. 3. Qualitative plot of the derivative of the center position describing the kink pinning process. Averages are taken in Eq. (79) as explained in the text.

To put things in a more simple (though less rigorous) form, we can also think of computing the average of  $\Phi$  in the rhs of Eq. (79) with the help of an *effective* Gaussian distribution for  $\zeta$  (the probability distribution for  $\zeta$  must necessarily be Gaussian-like), say

$$P_{\text{eff}}(\zeta) = \frac{1}{2\sqrt{\pi}\sigma_{\text{eff}}} \exp\left(-\frac{(\zeta - u_0 t)^2}{4\sigma_{\text{eff}}^2}\right), \quad (81)$$

with  $\sigma_{\text{eff}} \equiv D_{\text{eff}} t$ . We do not know  $\sigma_{\text{eff}}$  exactly, but we know that it must verify  $\sigma_{\text{eff}} \propto \hat{D}$ , it must increase with time, and it must become of order unity at  $t \sim \hat{D}^{-1}$ . With such a distribution, we have computed numerically the rhs of Eq. (79), and the result is shown in Fig. 3, where we have been put  $u_0 = 0.1$  and  $D_{\text{eff}} = \hat{D} = 2u_0$  (these parameters have been chosen only for the sake of greater clarity of the plot and not for any special physical reason). It can be appreciated how the kink suffers a slowing down process that ends with its pinning inside the noisy zone but near the boundary.

## VI. LINEAR THEORY OF RADIATION EVOLUTION

The last question we are going to solve in this work comes once more from numerics. We have observed<sup>32</sup> that kinks suffer an anomalous diffusion; to be precise, we can point out the following features:

(1) Anomalous diffusion occurs at times  $t > D^{-1}$ . With the words “anomalous diffusion” we mean that the kink center dispersion is not linear with  $t$  anymore; instead, it grows with  $t^\delta$ , the exponent  $\delta$  being  $1 < \delta < 2$ .

(2) The effect becomes weaker as the kink velocity approaches unity.

(3) Anomalous diffusion is accompanied by an exponential growth of the total energy contained in the system.

(4) Energy growth disappears if dissipation is taken into account.

As can be seen from Eqs. (35) and (36), together with the first mentioned characteristic, all these properties *cannot* be explained by means of an extrapolation of the adiabatic approximation for large times, and therefore we need another approach to understand the behavior of the stochastic  $\phi^4$  model for not so small times.

The most natural assumption in order to study these phenomena is that anomalous diffusion is caused by changes of the spectrum of waves propagating along the system, these changes resulting from the interaction of the noisy radiation generated by the kink with the fluctuating potential. To verify this hypothesis, it is necessary to consider the evolution of linear waves in the system.

As has been stated in Sec.III, due to the influence of the perturbation, a kink generates linear modes, with group velocities different from its own one. This fact leads to the spatial separation of the kink and its radiation. However, all the above described perturbative calculations hold only for small times; the exact threshold for them to be valid depends on noise strength, that being of the order of  $(2D)^{-1/2}$ . We are now trying to go a little further, and get a flavor of the subsequent evolution

of the radiation created at the early stages of the propagation. To this end, let us come back to the continuum equation (1). As we want to describe small excitations of the system, we can linearize it around the kink. If we do so, and subsequently we look for usual plane wave solutions of the form  $\phi(x, t) \equiv \phi(t)e^{iKx}$ , where  $K$  can be understood as the wave vector of a phonon emitted by the kink and is one of both  $K_\pm$  [see Eq. (51)], we arrive at

$$\ddot{\phi}(t) + [\Omega^2 + \xi(t)] \phi(t) = 0, \quad (82)$$

dots meaning now derivatives with respect to time, and  $\Omega$  being one of those in Eq. (54). As usual, we can rewrite Eq. (82) as two ordinary first-order equations,

$$\dot{\phi}(t) = \nu(t), \quad (83)$$

$$\dot{\nu}(t) = -\Omega^2 \phi(t) - \xi(t)\phi(t). \quad (84)$$

From these two equations we can go on obtaining the statistical properties of the amplitudes  $\phi(t)$  and  $\nu(t)$ . The system (83) and (84) is well known in statistical radiophysics,<sup>36</sup> and we believe that it is not necessary to explain in detail the calculations related to it but only to write down the most important intermediate steps. The main feature of the random differential equation (82) is the stochastic parametrical resonance. To deal with it, we now calculate the first two momenta of the stochastic processes  $\phi(t)$  and  $\nu(t)$ . By means of the Novikov theorem (68) (see analogous calculations in Sec.V), we can carry out the direct averaging of Eqs. (83) and (84), that yields

$$\langle \dot{\phi} \rangle = \langle \nu \rangle, \quad (85)$$

$$\langle \dot{\nu} \rangle = -\Omega^2 \langle \phi \rangle. \quad (86)$$

So, noise does not play any part at all in the evolution of the mean value of the radiation amplitude that happens to behave periodically,

$$\langle \phi \rangle(t) = \phi_+ e^{i\Omega t} + \phi_- e^{-i\Omega t}. \quad (87)$$

The amplitudes  $\phi_\pm$  must be determined from initial conditions, but we are not going to specify them now.

The lowest momenta of the radiation amplitude in which noise effects do appear are the second ones. The corresponding equations are

$$\frac{d\langle \phi^2 \rangle}{dt} = 2\langle \phi \nu \rangle, \quad (88)$$

$$\frac{d\langle \nu^2 \rangle}{dt} = -2\Omega^2 \langle \phi \nu \rangle + 2D\langle \phi^2 \rangle, \quad (89)$$

$$\frac{d\langle \phi \nu \rangle}{dt} = \langle \nu^2 \rangle - \Omega^2 \langle \phi^2 \rangle. \quad (90)$$

As is usual, we look for a solution in the form of a function proportional to  $\exp(\lambda t)$ , where  $\lambda$  is a constant. By so doing, the following secular equation arises from (88), (89), and (90) for the allowed values of  $\lambda$ :

$$\lambda^3 + 4\lambda\Omega^2 - 4D = 0. \quad (91)$$

All the roots of this equation can be found analytically. We are interested only on those having positive real parts. Taking into account that the largest positive  $\lambda$  plays the

role of an increment time related to the growth of the energy contained in the linear modes, we solve for this value and arrive at

$$\lambda = 2^{1/3}[(\alpha^{1/2} + D)^{1/3} - (\alpha^{1/2} - D)^{1/3}], \quad (92)$$

where

$$\alpha \equiv \frac{16}{27} \Omega^6 + D. \quad (93)$$

In particular, for high frequency phonons, this relationship goes asymptotically as

$$\lambda \simeq \frac{D}{\Omega^2}. \quad (94)$$

Thus, due to the stochastic parametrical resonance, the spectrum of linear noise initially generated by the presence of a kink in the system enriches itself in time with waves of long wavelength (low frequencies) much more effectively than with high frequency linear waves. This property allows us to give an explanation of all the items listed above concerning the anomalous diffusion of a kink.

Indeed, a kink produces radiation during its evolution in a medium with fluctuating parameters. This radiation goes outwards from the soliton because of the different group velocities, and correspondingly it consists of linear excitations of the system. As follows from Eq. (50), the power of the generation process is proportional to the factor  $(1 - v^2)$ , and subsequently it vanishes as  $v \rightarrow 1$ . Since its creation, the so generated linear waves parametrically excite themselves due to the interaction with the fluctuating potential [see Eqs. (88), (89), and (90)]; this happens to be important after an effective time of order  $\lambda$ , that goes to infinity when  $D \rightarrow 0$ . This means that at the moment  $t \sim D^{-1}$  the fluctuating potential interacts effectively not only with the kink but also with linear noise generated by the kink itself. However, the spectrum of this stochastic radiation is not constant, as it would be if it were formed by white noise. Instead, it decreases as  $\Omega \rightarrow \infty$ . Moreover, in view of Eq. (94), it has a power law character [one can expand  $\exp(\lambda t)$  in a power series if  $\Omega \gg \sqrt{D/\lambda}$ ]. Phenomenologically we can describe the influence of this linear noise by forgetting the radiation and substituting it by an effective additive noise in the stochastic  $\phi^4$  model, which leads to fluctuations in  $v(t)$  (according to the adiabatic approximation) and as a consequence to the non-Brownian diffusion of the kink.

Thus, collecting all our findings, we have the following qualitative treatment for the above mentioned properties:

- (1) The amplitude of the generated linear noise is much less than the stochastic part of the potential at times  $t < D^{-1}$ . It becomes essential as  $t$  approaches and exceeds  $D^{-1}$ .
- (2) This amplitude is proportional to a factor  $(1 - v^2)$ , and goes to zero for ultrarelativistic solitons.
- (3) The numerically discovered exponential growth of the energy is a feature coming from the stochastic parametrical resonance.

The same kind of arguments can be used to consider the effects of the introduction of a dissipative term in

the system. In fact, as is well known, dissipation leads to the appearance of a threshold for the parametrical resonance to take place<sup>36</sup> (this can be ensured by repeating the previous calculations taking the dissipative part into account), given by<sup>36</sup>

$$D = \alpha \Omega^2, \quad (95)$$

$\alpha$  now being the dissipative coefficient. Substituting in (95) the values  $\alpha = 0.1$  and  $D = 0.05$ , one standard numerical choice, we conclude that only waves of low frequency, with  $\Omega^2 < 0.5$  are unstable, while the rest of the linear spectrum does not undergo any parametrical pump at all. This might provide a qualitative treatment of item (4) mentioned at the beginning of this section.

To conclude this section, let us remark that, strictly speaking, we should have included nonlinearity in the preceding discussion because it becomes more and more important as energy increases. Actually, nonlinearity can inhibit or limit this growth of energy;<sup>37</sup> even more, nonlinearity leads to a change in the exponent of the power law dependence of the noise spectrum as well. Unfortunately, this kind of description of the complete nonlinear problem cannot be done in the framework of our method.

## VII. CONCLUSION

In this last section, we summarize the results obtained in this work and describe what kink dynamics is in the stochastic  $\phi^4$  model, as it appears to be from our analytical approach valid for weak perturbations (i.e., for noises that are sufficiently weak or act during small enough time). This model is not exactly integrable even if the fluctuating part of the nonlinear potential is absent. So, it is not possible to apply a general perturbative machinery<sup>5</sup> that extends to multisoliton solutions; however, such a theory can be constructed for the dynamics of a perturbed kink because of the remarkable fact that the linear problem corresponding to the bound state associated with a kink (nonzero vacuum) can be solved exactly. The necessary condition for this to hold is the smallness of the intensity of the fluctuations. In the particular case of the stochastic  $\phi^4$  model without dissipation, the perturbative theory allows us to conclude that fluctuations result in a Brownian diffusion of the kink at times  $t < D^{-1}$ . Only one of the parameters initially defining the kink is affected by noise, namely, its center position. The main feature of the perturbed  $\phi^4$  kink consists of the excitation of the internal mode by the stochastic term. This mode does not affect the kink speed; it is localized near the kink kernel, and describes fluctuations of its shape. The energy of the internal mode grows with time according a diffusive, square root law. Besides, potential fluctuations result in a continuous generation of linear modes, i.e., linear noise. This process takes place at a constant ratio, and waves' evolution can be satisfactorily described by their corresponding linear equation. Noise effects are most intensive on slow kinks; all corrections to the unperturbed kink become vanishingly small in the relativistic case, that is, as soliton speed goes to unity. Kink dynamics changes if the fluctuating

part of the potential occupies only one half of the space. Its behavior essentially depends on the value of the initial velocity: if its initial speed is sufficiently small, the kink gets pinned beyond, but very near, the boundary. We have been able to show that it is possible to account for this phenomenon within the framework of our perturbative calculations as well, computing the derivative of the mean center position and seeing that it goes to zero if the initial speed is small enough.

The way the stochastic  $\phi^4$  model behaves changes dramatically as we allow the perturbation to act further than small times. When  $t$  reaches values around  $D^{-1}$ , we are no longer allowed to consider the problem by means of a perturbative theory because radiation is not only being generated by the propagating soliton but is also being pumped directly by the fluctuating potential. The kink then undergoes a fractal Brownian motion,<sup>38</sup> and an exponential growth of the energy contained in the chain is observed. It is very likely that this is related to a new phenomenon taking place in the system, namely, the stochastic parametrical resonance in the long time evolution of the linear modes, i.e., the phonons emitted by the kink. It is important to stress that this is a specific feature of the system under consideration, whose linear version is an oscillator with fluctuating frequency. Such a treatment leads us to conclude that the numerically detected exponential behavior of the system energy, is due to the increasing of the energetic content of linear modes rather than that of the kink itself. Note that this effect cannot be explained by the linear increasing of the energy of the internal mode. Dissipation also enters this scheme very well. Thus, it explains the existence of a threshold in the process of fractal Brownian motion and energy increasing. At small times,  $t < D^{-1}$ , the action of

dissipation is trivial, and it results only in the decreasing of the kink speed.

Finally, we must stress that all our reasoning and calculations in Secs. V and VI had only a qualitative character. We cannot give an exact statistical description of the phenomenon of the kink pinning at boundaries nor an exact determination of the exponent of kink anomalous diffusion. These remain open questions, as well as the study of kink dynamics in a spatially inhomogeneous stochastic  $\phi^4$  model, the influence of the value of spatial correlation radius on the fractal motion of kinks, the relation of this process to the localization problem, and so on. In addition, a natural generalization of the model we present here could be given by the change  $\xi(t)(\phi - \phi^3) \mapsto \xi_1(t)\phi - \xi_3(t)\phi^3$ ,  $\xi_1(t)$  and  $\xi_3(t)$  being different random processes. This extension, that would include as particular cases our work and the one in Ref. 28, and that in a general situation might give rise to completely different behaviors of the kink kernel, is currently under study.<sup>39</sup> All these problems need a great amount of further work to achieve complete understanding.

## ACKNOWLEDGMENTS

We are grateful to A. R. Bishop and Yu. S. Kivshar for useful advice. We acknowledge partial financial support from the Dirección General de Investigación Científica y Técnica (DGICYT, Spain) through Project No. PB86-0005. V.V.K. wishes to thank the Universidad Complutense for hospitality during his stay in Madrid, where this work was done in part. A.S. was supported by the program "Formación de Personal Investigador" of the Ministerio de Educación y Ciencia of Spain.

<sup>1</sup>N. J. Zabusky, J. Comp. Phys. **43**, 195 (1981).

<sup>2</sup>M. Remoissenet, in *Proceedings of the Les Houches Winter School in Theoretical Physics on Partially Integrable Nonlinear Equations and their Physical Applications* (Dordrecht, The Netherlands, 1990).

<sup>3</sup>*Solitons in Condensed Matter Physics*, Springer Series in Solid State Sciences, Vol. 8, edited by A. R. Bishop and T. Schneider (Springer-Verlag, Berlin, 1978).

<sup>4</sup>A. R. Bishop, J. A. Krumhansl, and S. E. Trullinger, Physica D **1**, 1 (1980).

<sup>5</sup>Yu. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).

<sup>6</sup>S. Aubry, J. Chem. Phys. **62**, 3217 (1975); **64**, 3392 (1976).

<sup>7</sup>J. A. Krumhansl and J. R. Schrieffer, Phys. Rev. B **11**, 3535 (1975).

<sup>8</sup>M. J. Rice, Phys. Lett. **71A**, 153 (1979); M. J. Rice and J. Timonen, *ibid.* **73A**, 369 (1979).

<sup>9</sup>F. Guinea, Phys. Rev. B **30**, 1884 (1984).

<sup>10</sup>M. B. Fogel, S. E. Trullinger, A. R. Bishop, and J. A. Krumhansl, Phys. Rev. B **15**, 1578 (1977).

<sup>11</sup>G. Eilenberger, Z. Phys. B **27**, 199 (1977).

<sup>12</sup>D. W. McLaughlin and A. C. Scott, Phys. Rev. A **18**, 1652 (1978); in *Solitons in Action*, edited by K. Longrenn and A. C. Scott (Academic, New York, 1978).

<sup>13</sup>R. Boesch, P. Stancioff, and C. Willis, Phys. Rev. B **38**,

6713 (1988).

<sup>14</sup>N. Theodorakopoulos, Z. Phys. B **33**, 385 (1979).

<sup>15</sup>Y. Wada and J. R. Schrieffer, Phys. Rev. B **18**, 3897 (1978).

<sup>16</sup>D. K. Campbell, J. F. Schonfeld, and C. A. Wingate, Physica D **9**, 1 (1983), and references therein.

<sup>17</sup>M. J. Ablowitz, M. D. Kruskal, and J. R. Ladik, SIAM J. Appl. Math. **36**, 478 (1979).

<sup>18</sup>N. Flytzanis, St. Pnevmatikos, and M. Remoissenet, J. Phys. C **18**, 4603 (1985).

<sup>19</sup>J. A. Combs and S. Yip, Phys. Rev. B **28**, 6873 (1983); **29**, 438 (1984); C. Kunz and J. A. Combs, *ibid.* **31**, 527 (1985).

<sup>20</sup>J. F. Currie, S. E. Trullinger, A. R. Bishop, and J. A. Krumhansl, Phys. Rev. B **15**, 5567 (1977).

<sup>21</sup>St. Pnevmatikos, N. Flytzanis, and A. R. Bishop, J. Phys. C **20**, 2829 (1987).

<sup>22</sup>T. Fraggis, St. Pnevmatikos, and E. N. Economou, Phys. Lett. **142A**, 361 (1989).

<sup>23</sup>F. G. Bass, Yu. S. Kivshar, V. V. Konotop, and Yu. A. Sinitsyn, Phys. Rep. **157** 63 (1988).

<sup>24</sup>M. A. Collins, A. Blumen, J. F. Currie, and J. Ross, Phys. Rev. B **19**, 3630 (1979); J. F. Currie, A. Blumen, M. A. Collins, and J. Ross, *ibid.* **19**, 3645 (1979).

<sup>25</sup>A. A. Abdumalikov, F. Kh. Abdullaev, and S. Stamenkovič, Phys. Status Solidi B **120**, K33 (1983); F. Kh. Abdullaev and S. A. Darmanyan, *ibid.* **158**, K201 (1990).

- <sup>26</sup>F. G. Bass, V. V. Konotop, and Yu. A. Sinitsyn, *Izv. Vyssh. Uchebn. Zaved., Radiofiz.* **27**, 718 (1984) [*Sov. J. Radiophys. Electron.* **27**, 491 (1984)]
- <sup>27</sup>F. G. Bass, V. V. Konotop, and Yu. A. Sinitsyn, *Izv. Vyssh. Uchebn. Zaved., Radiofiz.* **29**, 921 (1986) [*Sov. J. Radiophys. Electron.* **29**, 708 (1987)].
- <sup>28</sup>M. J. Rodríguez-Plaza and L. Vázquez, *Phys. Rev. B* **41**, 11 437 (1990).
- <sup>29</sup>J. M. R. Parrondo, M. Mañas, and F. J. de la Rubia, *J. Phys. A* **23**, 2363 (1990).
- <sup>30</sup>Yu. S. Kivshar, S. A. Gredeskul, A. Sánchez, and L. Vázquez, *Phys. Rev. Lett.* **64**, 1693 (1990).
- <sup>31</sup>A. Sánchez and L. Vázquez, *Phys. Lett.* **152A**, 184 (1991).
- <sup>32</sup>A. Sánchez, L. Vázquez, and V. V. Konotop, *Phys. Rev. A* **44**, 1086 (1991).
- <sup>33</sup>R. J. Flesch and S. E. Trullinger, *J. Math. Phys.* **28**, 1619 (1987), and references therein.
- <sup>34</sup>P. Morse and H. Feshbach, *Methods of Mathematical Physics* (McGraw-Hill, New York, 1953).
- <sup>35</sup>E. A. Novikov, *Zh. Eksp. Teor. Fiz.* **47**, 1919 (1964) [*Sov. Phys. JETP* **20**, 1290 (1965)].
- <sup>36</sup>V. I. Klyatskin, *Stochastic Equations and Waves in Randomly Inhomogeneous Media* (Nauka, Moscow, 1980) (in Russian).
- <sup>37</sup>V. A. Kagalovskij and V. V. Konotop, *Izv. Vyssh. Uchebn. Zaved., Radiofiz.* **30**, 553 (1987).
- <sup>38</sup>B. B. Mandelbrot and J. W. van Ness, *SIAM Rev.* **10**, 422 (1968) is the paper where the concept of fractal Brownian motion was introduced. See also the corresponding chapters of the following monographs: J. Feder, *Fractals* (Plenum, New York, 1988); T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989); and K. J. Falconer, *Fractal Geometry* (Wiley, Sussex, 1990).
- <sup>39</sup>V. V. Konotop, A. Sánchez, and L. Vázquez (unpublished).